Consistent Sphere Reductions and Universality of the Coulomb Branch in the Domain-Wall/QFT Correspondence

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ABSTRACT

We prove that any D-dimensional theory comprising gravity, an antisymmetric n-index field strength and a dilaton can be consistently reduced on S^n in a truncation in which just n scalar fields and the metric are retained in $(D-n)$-dimensions, provided only that the strength of the couping of the dilaton to the field strength is appropriately chosen. A consistent reduction can then be performed for $n \leq 5$; with D being arbitrary when $n \leq 3$, whilst $D \leq 11$ for $n = 4$ and $D \leq 10$ for $n = 5$. (Or, by Hodge dualisation, n can be replaced by $(D-n)$ in these conditions.) We obtain the lower dimensional scalar potentials and construct associated domain wall solutions. We use the consistent reduction Ansatz to lift domain-wall solutions in the $(D-n)$-dimensional theory back to D dimensions, where we show that they become certain continuous distributions of $(D-n-2)$-branes. We also examine the spectrum for a minimally-coupled scalar field in the domain-wall background, showing that it has a universal structure characterised completely by the dimension n of the compactifying sphere.

1 Introduction

The ability to embed a lower-dimensional theory in a higher-dimensional one has proved to be an extremely useful one in string theory. One can, for example, re-interpret lower-dimensional p-brane solitons as solutions of the ten-dimensional string, or eleven-dimensional M-theory. A crucial aspect of this picture is that the Kaluza-Klein reduction must be a *consistent* one, in the sense that all solutions of the lower-dimensional theory must also be solutions of the original higher-dimensional theory. This consistency is guaranteed in a standard toroidal reduction, but it is far less clear-cut when a reduction on a manifold such as a sphere is considered.

Kaluza-Klein reductions on spheres are of great interest in the framework of string theory, because they can give rise to lower-dimensional gauged supergravities that are relevant for discussing the AdS/CFT correspondence [1, 2, 3]. The generic structure of these gauged supergravities comprises gravity coupled to a set of Yang-Mills gauge fields, and a set of scalar fields with a non-trivial potential, together, possibly, with additional antisymmetric tensor fields. A particular class of solution that can be studied is extremal domain walls, which can be viewed as charged black holes or black p-branes in the gauged theory, in the extremal limit for which the electric or magnetic charges actually vanish. Thus these solutions are supported entirely by the metric and certain scalar fields within the gauged supergravity.

It therefore becomes of interest to study the circumstances under which a higher-dimensional theory can admit a consistent n-sphere reduction in which just gravity and appropriate scalar fields are retained. In some cases this may be viewed as a subset of a larger consistent reduction of a gauged supergravity, in which the starting point is supergravity in ten or eleven dimensions. However, the question can also be posed in a more general framework, where the starting point need not necessarily even be a supersymmetric theory.

Before discussing the possible new cases, let us review what is known at present. It is natural, when considering an n-sphere reduction, to try to retain all the $SO(n + 1)$ Yang-Mills fields as part of the consistent reduction. Usually, however, this is not possible. It was recently shown in [4, 5] that the cases where this can be done, starting from a D-dimensional theory of gravity, n-form field strength and dilaton, are as follows. One can start with $(D, n) = (11, 4)$, and reduce on S^4 or S^7 to seven or four dimensions respectively; another possibility is to start from $(D, n) = (10, 5)$, and reduce on S^5 to five dimensions. In these cases, the system has no dilaton. Including a dilaton, with a specific coupling, one can

also start with $n = 3$ and D arbitrary, reducing on S^3 or S^{D-3}; or finally one can start with $n = 2$ and D arbitrary, and reduce on S^2. In all cases one must also include scalar fields T_{ij} in the reduction Ansatz, corresponding to the coset $SL(n+1, \mathbb{R})/SO(n+1)$. Additionally, for the S^4 reduction of $D = 11$ one must include five 3-form field strengths in the Ansatz, while for the S^7 reduction one must also include 35 more pseudoscalars.[1] Finally, for the S^3 reduction in the $n = 3$ case, one must include a 3-form field strength in the reduction Ansatz.

Our statement of the possible consistent reductions first specified that the $SO(n+1)$ Yang-Mills gauge fields were to be included, and then we listed the additional fields that would be needed for consistency. Another way of phrasing the question is to specify which scalar fields will be included in the reduction Ansatz. In fact if we want to include all the scalars T_{ij} of the $SL(n+1, \mathbb{R})/SO(n+1)$ coset, the list of cases where consistent reductions are possible will be the same as the above. The reason for this is that once all the scalars T_{ij} are present, they will act as sources for the Yang-Mills gauge fields, and so it would be inconsistent to omit the Yang-Mills fields. However, if we settle for a reduction in which fewer scalars are retained, it becomes possible to omit the Yang-Mills fields and this opens up some further possibilities for consistent reductions, which we shall explore in this paper. These reductions with scalars but no gauge fields will be sufficient for the purpose of constructing the extremal domain-wall solutions in the lower dimension, and then lifting them back to the higher dimension.

As mentioned above, if one includes the full set of scalars T_{ij} in a truncation then they will give rise to source terms that require the Yang-Mills fields to be non-zero. Specifically, the source currents are of the form $T_{k[i}^{-1} \partial_\mu T_{j]k}$, in the adjoint of $SO(n+1)$. If we make a truncation where only the diagonal scalar fields are retained,

$$T_{ij} = \text{diag}(X_1, X_2, \ldots X_{n+1}), \tag{1}$$

then the currents $T_{k[i}^{-1} \partial_\mu T_{j]k}$ will be zero, and thus there is no longer any necessity to include the gauge fields in a consistent truncation. This actually allows a somewhat extended set of (D, n) values for which consistent reductions can be achieved, which includes cases that would not allow consistent reductions with $SO(n+1)$ gauge fields. The allowed cases are detailed below.

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[1] In fact the consistent reductions from $D = 11$ require, in addition, the inclusion of the FFA in the Lagrangian that arises in $D = 11$ supergravity. In the S^5 reduction from $D = 10$, it is necessary to impose the requirement of self-duality on the 5-form field strength.

In section 2 we construct an Ansatz for the n-sphere reduction of the a D-dimensional theory of gravity, an n-form field strength, and a dilaton, in which the lower-dimensional fields comprise just gravity and the diagonal scalar fields given by (1). We obtain a complete proof of the consistency of this Kaluza-Klein reduction, showing that it works in all cases where the strength of the coupling of the dilaton to the n-form in D dimensions is appropriate. This requirement on the coupling is a rather stringent one, and the allowable cases turn out to be $\{n = 5, D \leq 10\}$; $\{n = 4, D \leq 11\}$; and $n \leq 3$ with D arbitrary, for $n \leq D/2$.

In section 3 we construct $(n+1)$-parameter extremal domain-wall solutions in the lower-dimensional theories of gravity plus scalar fields, and then make use of the reduction Ansatz derived in section 2 in order to lift these solutions back to the original D-dimensional theory. We show that in the higher dimension the lifted solutions admit an interpretation as continuous distributions of $(D - n - 2)$-branes. We discuss and obtain the distribution functions. We obtain the metric of the distributed branes in the dual frame, and show that the structure of these metrics depends only on the dimension n of the internal sphere, but is independent of D. In particular, the metric in the dual frame becomes asymptotically AdS$\times S^n$ for $n \neq 3$, and Minkowski$\times S^3$ for $n = 3$.

In section 4 we analyse the spectrum of excitations of a minimally-coupled scalar in the background of the $(D - n)$-dimensional domain-wall solution, showing that it has a universal structure that is characterised by the dimension n of the internal sphere used in the dimensional reduction. In the case of the vacuum solutions, where the $(n+1)$ parameters in the general solutions are all set to zero, the scalar wave equation can be solved explicitly, allowing a study of the singularity structure. We also analyse the Schrödinger potentials for generic cases, allowing us to determine the structures of the spectra in the various cases.

In an appendix, we show that a single-charge rotating p-brane in a generic dimension can be dimensionally reduced on the internal (distorted) n-sphere to give rise to domain-wall black holes with $[(n + 1)/2]$ electric $U(1)$ charges. In the extremal limit, the gauge fields vanish and the balck hole becomes a domain wall that is contained within the set of solutions obtained in this paper.

2   Kaluza-Klein sphere reduction

Single-charge p-branes in supergravity theories in D dimensions can be classified as solutions of the theory described by the Einstein-Hilbert action coupled to a dilaton and an n-form

field strength,

$$\mathcal{L}_D = \hat{R} \,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\hat{\phi} \wedge d\hat{\phi} - \tfrac{1}{2}\, e^{a\hat{\phi}}\, \hat{*}\hat{F}_{(n)} \wedge \hat{F}_{(n)}\,, \tag{2}$$

where the constant a is given by [6]

$$a^2 = 4 - \frac{2(n-1)(D-n-1)}{D-2}\,. \tag{3}$$

The requirement that a be real puts a strong condition on the possible values for n, bearing in mind that we must have $n \le D$, and in fact we can always choose a dualisation for $F_{(n)}$ for which $n \le D/2$. From (3), it then follows that the maximum value for $n \le D/2$ is 5. Whe $n = 5$, the maximal dimension is $D = 10$, corresponding to the self-dual 5-form in the type IIB theory. For $n = 4$, the maximal dimension is $D = 11$, corresponding to 11-dimensional supergravity. In both cases, the constant a vanishes. For $n = 0, 1, 2, 3$, the dimension D can be arbitrary. Note that for a given n satisfying (3), $n' = D - n$ satisfies it too. To summarise, the allowed possibilities are

$$
\begin{aligned}
n = 5\,, \quad D - 5: & \quad D \le 10 \\
n = 4\,, \quad D - 4: & \quad D \ge 11 \\
n = 0, 1, 2, 3\,, \quad D, D-1, D-2, D-3: & \quad D \quad \text{arbitrary}\,.
\end{aligned}
\tag{4}
$$

Note that these results come from the requirement only that a must be real. If in addition we require that the Lagrangian must be associated with a supersymmetric theory, we get the further restriction that the dimension D must be less than or equal to eleven or ten.

The p-branes for which the first term on the right-hand-side of (3) is 4 can be viewed as the basic building blocks for p-brane solitons. The p-branes with values other than 4, (usually $4/N$ with N an integer) can be viewed as bound states or intersections of these building blocks. For example, for $D = 11$ and $D = 10$, our discussion applies to M-branes, the NS-NS string and 5-brane and all the D-branes.

We shall now consider the Kaluza-Klein dimensional reduction of the Lagrangian (2) on S^n. (The discussion of the reduction instead on S^{D-n} can be handled by dualising the n-form field strength to a $(D - n)$-form.) In general, such a reduction is inconsistent if we keep all the massless fields. It was shown [4], however, that for $n = 2$ and $n = 3$ the reduction is always consistent, provided that (3) is satisfied. For $n = 5$ and $n = 4$, the reduction is consistent only if additional conditions are satisfied, namely self-duality of the 5-form in $D = 10$, and the addition of an FFA term in $D = 11$ for the $n = 4$ case.

In this paper, we shall truncate further to a subset of the massless fields, corresponding to "diagonal inhomogeneous distortions" of the internal S^n metric. By this, we mean that

we canonically embed the sphere S^n in $n + 1$ dimensional Euclidean space. The round S^n metric is given by $d\mu_i d\mu_i$, where μ_i are Euclidean coordinates satisfying the unit-length constraint $\mu_i \mu_i = 1$. The diagonal inhomogeneous distortion of the sphere is then achieved by introducing $(n + 1)$ scalars X_i, and scaling the coordinate differentials as follows:

$$ds_n^2 = \sum_i X_i^{-1} (d\mu_i)^2 . \tag{5}$$

We shall show that for this subset of fields, the Kaluza-Klein reduction is consistent for any of the D and n values listed in (4), provided that (3) is satisfied.

We find that the Kaluza-Klein reduction Ansatz is given by

$$d\hat{s}_D^2 = Y^{\frac{1}{D-2}} \left(\Delta^{\frac{n-1}{D-2}} ds_{D-n}^2 + g^{-2} \Delta^{-\frac{(D-n-1)}{(D-2)}} \sum_{i=1}^{n+1} X_i^{-1} (d\mu_i)^2 \right),$$

$$e^{-\frac{2}{a}\hat{\phi}} = \Delta^{-1} Y^{\frac{2(D-n-1)}{a^2(D-2)}}, \tag{6}$$

$$\hat{F}_{(n)} = g^{-n+1} \Delta^{-2} U W + g^{-n+1} \partial_\nu \left(\frac{X_i \mu^i}{\Delta} \right) dx^\nu \wedge Z_i .$$

where

$$\mu_i \mu_i = 1, \quad Y = \prod X_i, \quad \Delta = \sum X_i \mu_i^2, \quad U = 2 \sum_i X_i^2 \mu_i^2 - \Delta \sum_i X_i . \tag{7}$$

The quantities W and Z_i are respectively the volume-form on the n-sphere, and a certain $(n-1)$-form on the n-sphere:

$$W = \frac{1}{n!} \epsilon_{i j_1 \cdots j_n} \mu^i d\mu^{j_1} \wedge \cdots \wedge d\mu^{j_n}, \tag{8}$$

$$Z_i = \frac{1}{(n-1)!} \epsilon_{i j_1 \cdots j_n} \mu^{j_1} d\mu^{j_2} \wedge \cdots \wedge d\mu^{j_n} .$$

We find after some algebra that the dual of the field strength $F_{(n)}$ is given by

$$e^{a\hat{\phi}} \hat{*} F_{(n)} = g U \epsilon_{D-n} + \frac{1}{2g} X_i^{-1} * dX_i \wedge d(\mu_i^2) . \tag{9}$$

We can then substitute the Ansatz into higher dimensional equations of motion. First, we can verify that the Ansatz for $\hat{F}_{(n)}$ in (6) satisfies the Bianchi identity $d\hat{F}_{(n)} = 0$. Next, we look at the equations of motion for the field strength $\hat{F}_{(n)}$ and the dilaton $\hat{\phi}$:

$$d\left(e^{a\hat{\phi}} \hat{*} \hat{F}_{(n)} \right) = 0,$$

$$(-1)^D d\hat{*}d\hat{\phi} = -a e^{a\hat{\phi}} \hat{*} \hat{F}_n \wedge \hat{F}_{(n)} . \tag{10}$$

After a considerable amount of algebra, we find that the Ansatz yields a consistent dimensional reduction of these D-dimensional equations to give the following $(D-n)$-dimensional

equations for the scalar fields:

$$(-1)^{D-n} d(\widetilde{X}_i^{-1} *d\widetilde{X}_i) = -2g^2 Y^{\frac{2}{n+1}} \left[2\widetilde{X}_i^2 - \widetilde{X}_i \sum_j \widetilde{X}_j - \tfrac{2}{n+1} \sum_j \widetilde{X}_j^2 + \tfrac{1}{n+1} (\sum_j \widetilde{X}_j)^2 \right] \epsilon_{D-n} ,$$

$$(-1)^{D-n} \tfrac{2(D-n-2)}{(D-2)\, a^2} d(Y^{-1} *dY) = -V \, \epsilon_{D-n} . \tag{11}$$

Here, we have defined the rescaled fields $\widetilde{X}_i$ by

$$X_i = Y^{\frac{1}{n+1}} \, \widetilde{X}_i , \tag{12}$$

so that $\prod_i \widetilde{X}_i = 1$, and the potential V is defined by

$$V \equiv \tfrac{1}{2} g^2 \left(2 \sum_i X_i^2 - (\sum_i X_i)^2 \right) = \tfrac{1}{2} g^2 \, Y^{\frac{2}{n+1}} \left(2 \sum_i \widetilde{X}_i^2 - (\sum_i \widetilde{X}_i)^2 \right) . \tag{13}$$

Finally, to check the higher-dimensional Einstein equations, we need first to calculate the Ricci tensor for the metric in (6). This is most easily done by noting that it is conformally related to the metric

$$d\bar{s}_D^2 = \Delta^p \, ds_{D-n}^2 + \Delta^{-q} \sum_i X_i^{-1} \, (d\mu_i)^2 , \tag{14}$$

with

$$d\hat{s}_D^2 = e^{2f} \, d\bar{s}_D^2 , \tag{15}$$

where we have defined

$$e^{2f} = Y^{\frac{1}{D-2}} , \qquad p = \frac{n-1}{D-2} , \qquad q = \frac{D-n-1}{D-2} . \tag{16}$$

It is easy to establish the standard result that the coordinate-frame components of the Ricci tensor $\hat{R}_{MN}$ for the metric $d\hat{s}_D^2$ are related to the coordinate-frame components $\bar{R}_{MN}$ for the metric $d\bar{s}_D^2$ by

$$\hat{R}_{MN} = \bar{R}_{MN} + (D-2)\Big(\partial_M f \, \partial_N f - \bar{\nabla}_M \, \partial_N \, f - \bar{g}^{PQ} \, (\partial_P f)(\partial_Q f) \, \bar{g}_{MN} \Big) - \bar{\Box} f \, \bar{g}_{MN} . \tag{17}$$

Results for the Ricci tensor for certain metrics of the form (14) were derived in [7], and with minor modifications they can be carried over to our present case. They were obtained in a basis where one of the $(n+1)$ coordinates μ_i, say μ_0, is expressed in terms of the n remaining ones μ_α by using the relation $\mu_i \, \mu_i = 1$. Thus the components $g_{\alpha\beta}$ of the distorted n-sphere metric (5), and its inverse, are given by

$$\begin{aligned}
g_{\alpha\beta} &= X_\alpha \, \delta_{\alpha\beta} + X_0^{-1} \, \hat{\mu}_\alpha \, \hat{\mu}_\beta , \\
g^{\alpha\beta} &= X_\alpha \, \delta_{\alpha\beta} - \Delta^{-1} \, X_\alpha \, X_\beta \, \mu_\alpha \, \mu_b ,
\end{aligned} \tag{18}$$

where in the first line we are writing $\hat{\mu}_\alpha = \mu_\alpha/\mu_0$. We refer to [7] for many of the details of the curvature calculations. Combining these results with (17), we obtain, after extensive algebraic manipulations, the following expressions for the lower-dimensional spacetime, internal and mixed components of the D-dimensional Ricci tensor:

$$
\begin{aligned}
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \tfrac{(n-1)(D-n-1)}{4(D-2)}\,\Delta^{-2}\,\partial_\mu\Delta\,\partial_\nu\Delta - \tfrac{1}{2}p\,\Delta^{-1}\,\Box\Delta\,g_{\mu\nu} + \tfrac{1}{2}p\,\Delta^{-2}\,\partial_\lambda\Delta\,\partial^\lambda\Delta\,g_{\mu\nu} \\
&\quad - \tfrac{1}{4}X_i^{-2}\,\partial_\mu X_i\,\partial_\nu X_i + \tfrac{1}{2}\Delta^{-1}\,X_i^{-1}\,\mu_i^2\,\partial_\mu X_i\,\partial_\nu X_i + \tfrac{1}{4(D-2)}\,Y^{-2}\,\partial_\mu Y\,\partial_\nu Y \\
&\quad - \tfrac{1}{4}q\,\Delta^{-1}\,(\partial_\mu\Delta\,\partial_\nu Y + \partial_\nu\Delta\,\partial_\mu Y) - \tfrac{1}{2(D-2)}\,\Box\log Y\,g_{\mu\nu} \\
&\quad - p\left(\sum_i X_i^2 - \Delta^{-1}\,X_i^2\,\mu_i^2\sum_j X_j + 2\Delta^{-2}\,(X_i^2\,\mu_i^2)^2 - 2\Delta^{-1}\,X_i^3\,\mu_i^2\right)g_{\mu\nu}\,, \\
\hat{R}_{\alpha\beta} &= R_{\alpha\beta} + \tfrac{1}{2}q\,g_{\alpha\beta}\,\Delta^{-2}\,\Box\Delta - \tfrac{1}{2}q\,g_{\alpha\beta}\,\Delta^{-3}\,\partial_\lambda\Delta\,\partial^\lambda\Delta - \tfrac{1}{2}\Delta^{-1}\,\Box\,g_{\alpha\beta} \qquad (19) \\
&\quad + \tfrac{1}{2}\Delta^{-1}\,g^{\gamma\delta}\,\partial_\lambda g_{\alpha\gamma}\,\partial^\lambda g_{\beta\delta} - \tfrac{1}{4}\Delta^{-2}\,\partial_\alpha\Delta\,\partial_\beta\Delta - \tfrac{1}{2}\Delta^{-1}\,\nabla_\alpha\partial_\beta\Delta \\
&\quad - \tfrac{1}{4}q\,g_{\alpha\beta}\,\Delta^{-2}\,\partial_\gamma\Delta\,\partial^\gamma\Delta + \tfrac{1}{2}q\,g_{\alpha\beta}\,\Delta^{-1}\,\nabla_\gamma\partial^\gamma\Delta - \tfrac{1}{2(D-2)}\,\Delta^{-1}\,\Box\log Y\,g_{\alpha\beta}\,, \\
\hat{R}_{\alpha\mu} &= -\tfrac{1}{2}\Delta^{-2}\,U\,(X_\alpha^{-1}\,\partial_\mu X_\alpha - X_0^{-1}\,\partial_\mu X_0)\,\mu_\alpha + \tfrac{1}{8}a^2\,\Delta^{-2}\,\partial_\mu\Delta\,\partial_a\Delta - \tfrac{1}{4}q\,\Delta^{-1}\,Y^{-1}\,\partial_\mu Y\,\partial_\alpha\Delta\,.
\end{aligned}
$$

Note that here we are using a "generalised" summation convention in which summations over the i index, where not otherwise indicated, are understood. The $\Box$ operator denotes the d'Alembertian calculated in the lower-dimensional metric $g_{\mu\nu}$, and $R_{\alpha\beta}$ denotes the Ricci tensor of the internal metric (i.e. the Ricci tensor for the metric (5), with the X_i are treated as parameters independent of the internal coordinates).

The D-dimensional Einstein equation reads $\hat{R}_{MN} = \hat{S}_{MN}$, where

$$
\hat{S}_{MN} = \tfrac{1}{2}\partial_M\hat{\phi}\,\partial_N\hat{\phi} + \frac{e^{a\hat{\phi}}}{2(D-n-1)!}\left(\hat{F}_{MN}^2 - \frac{D-n-3}{(D-n)(D-n-1)}\,\hat{F}^2\,\hat{g}_{MN}\right). \qquad (20)
$$

After some algebra we find that $\hat{S}_{MN}$ is given by

$$
\begin{aligned}
\hat{S}_{\mu\nu} &= \tfrac{1}{2}\Delta^{-1}\,\mu_i^2\,X_i^{-1}\,\partial_\mu X_i\,\partial_\nu X_i - \tfrac{(n-1)(D-n-1)}{4(D-2)}\,\Delta^{-2}\,\partial_\mu\Delta\,\partial_\nu\Delta + \tfrac{(D-n-1)^2}{2a^2\,(D-2)^2}\,Y^{-2}\,\partial_\mu Y\,\partial_\nu Y \\
&\quad - \tfrac{1}{4}q\,\Delta^{-1}\,Y^{-1}\,(\partial_\mu\Delta\,\partial_\nu Y + \partial_\nu\Delta\,\partial_\mu Y) \\
&\quad - \tfrac{1}{2}p\,\Delta^{-2}\left(U^2 - \partial_\lambda\Delta\,\partial^\lambda\Delta + \Delta\mu_i^2\,X_i^{-1}\,\partial_\lambda X_i\,\partial^\lambda X_i\right)g_{\mu\nu}\,, \\
\hat{S}_{\alpha\beta} &= \tfrac{1}{2}q\,\Delta^{-3}\,U^2\,g_{\alpha\beta} + \tfrac{1}{2}q\,\Delta^{-2}\,g_{\alpha\beta}\,X_i^{-1}\,\mu_i^2\,\partial_\lambda X_i\,\partial^\lambda X_i - \tfrac{1}{2}q\,\Delta^{-3}\,\partial_\lambda\Delta\,\partial^\lambda\Delta\,g_{\alpha\beta} \\
&\quad - \tfrac{1}{2}\Delta^{-2}\,(X_\alpha^{-1}\,\partial_\lambda X_\alpha - X_0^{-1}\,\partial_\lambda X_0)(X_\beta^{-1}\,\partial^\lambda X_\beta - X_0^{-1}\,\partial^\lambda X_0)\,\mu_\alpha\,\mu_\beta \\
&\quad + \tfrac{1}{8}a^2\,\Delta^{-2}\,\partial_\alpha\Delta\,\partial_\beta\Delta\,, \qquad (21) \\
\hat{S}_{\alpha\mu} &= -\tfrac{1}{2}\Delta^{-2}\,U\,(X_\alpha^{-1}\,\partial_\mu X_\alpha - X_0^{-1}\,\partial_\mu X_0)\,\mu_\alpha + \tfrac{1}{8}a^2\,\partial_\mu\Delta\,\partial_\alpha\Delta - \tfrac{1}{4}q\,\Delta^{-1}\,Y^{-1}\,\partial_\mu Y\,\partial_\alpha\Delta\,.
\end{aligned}
$$

After making use of the already-established equations of motion for the scalar fields, we eventually find after considerable further algebra that the $\hat{R}_{\mu\nu} = \hat{S}_{\mu\nu}$ components of the

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higher-dimensional Einstein equation imply

$$R_{\mu\nu} = \tfrac{1}{4}\widetilde{X}_i^{-2}\,\partial_\mu\widetilde{X}_i\,\partial_\nu\widetilde{X}_i + \frac{2(D-n-2)}{(D-2)(n+1)\,a^2}\,Y^{-2}\,\partial_\mu Y\,\partial_\nu Y + \frac{1}{D-n-2}\,V\,g_{\mu\nu}\,.\qquad(22)$$

The full system of $(D-n)$-dimensional equations of motion can therefore be derived from the Lagrangian

$$\mathcal{L} = R*\mathbb{1} - \frac{2(D-n-2)}{(n+1)(D-2)\,a^2}\,Y^{-2}*dY\wedge dY - \tfrac{1}{4}\sum_i \widetilde{X}_i^{-2}*d\widetilde{X}_i\wedge d\widetilde{X}_i - V*\mathbb{1}\,.\qquad(23)$$

It remains to check the consistency of the other components of the D-dimensional Einstein equations. After making use of the lower-dimensional equations of motion for the scalar fields, we find that the internal components $\hat{R}_{\alpha\beta}$ of the higher-dimensional Ricci tensor agree precisely with the expression for $\hat{S}_{\alpha\beta}$ that follows from substituting the Ansäze for $\hat{F}_{(n)}$ and $\hat{\phi}$, given in (6), into (21). Again, we have made extensive use of formulae derived in [7], appropriately modified to the case under consideration here. Finally, we note that the mixed components $\hat{R}_{\alpha\mu}$ in (19) agree precisely with the mixed components of $\hat{S}_{\alpha\mu}$ given in (21).

With these calculations we have now obtained a complete and explicit proof that the Ansatz (6) yields a consistent Kaluza-Klein n-sphere reduction of the D-dimensional theory described by (2), with the lower-dimensional fields appearing in the Ansatz satisfying the equations of motion that follow from the $(D-n)$-dimensional Lagrangian (23).

3 Domain walls as distributions of p-branes

We find that the d-dimensional gravity/scalar Lagrangian (23) admits a domain wall solution, given by

$$\begin{aligned}
ds_d^2 &= (gr)^{\frac{a^2(D-2)}{2(d-2)}}\left((gr)^{n-3}\,h^{\frac{1}{2(d-2)}}\,dx^\mu dx_\mu + h^{-\frac{d-3}{2(d-2)}}\,\frac{dr^2}{g^2r^2}\right),\\
X_i &= (gr)^{\frac{a^2(D-2)}{4(d-2)}}\,h^{\frac{(d-3)}{4(d-2)}}\,H_i^{-1}\,,
\end{aligned}\qquad(24)$$

where

$$h \equiv \prod_{i=1}^{n+1} H_i\,,\qquad H_i = 1 + \frac{\ell_i^2}{r^2}\,.\qquad(25)$$

In fact there is a redundancy in the paramtrisation of these solutions, which can be seen as follows. We make the following transformation of the radial coordinate,

$$r^2 = R^2 - L^2\,,\qquad(26)$$

where L is a constant, and define new quantities as follows:

$$\widetilde{H}_i \equiv 1 + \frac{\tilde{\ell}_i^2}{R^2}\,, \qquad \tilde{h} \equiv \prod_{i=1}^{n+1} \widetilde{H}_i\,, \qquad \tilde{\ell}_i^2 \equiv \ell_i^2 - L^2\,. \tag{27}$$

After straightforward calculations, we find that the solution (24) becomes

$$
\begin{aligned}
ds_d^2 &= (gR)^{\frac{a^2(D-2)}{2(d-2)}} \left((gR)^{n-3}\, \tilde{h}^{\frac{1}{2(d-2)}}\, dx^\mu dx_\mu + \tilde{h}^{-\frac{d-3}{2(d-2)}} \frac{dR^2}{g^2 R^2} \right)\,, \\
X_i &= (gR)^{\frac{a^2(D-2)}{4(d-2)}}\, \tilde{h}^{\frac{(d-3)}{4(d-2)}}\, \widetilde{H}_i^{-1}\,,
\end{aligned} \tag{28}
$$

This is identical in form to the original solution (24), but with the redefined functions given in (27). Let us suppose that, without loss of generality, the parameters ℓ_i are ordered so that $\ell_1^2 \geq \ell_2^2 \geq \cdots \geq \ell_{n+1}^2$. If we choose the constant L in the coordinate transformation (26) to be equal to ℓ_{n+1}, then we see that the original solution with $(n+1)$ parameters ℓ_i (with $1 \leq i \leq n+1$) is really nothing but a solution with only n parameters $\tilde{\ell}_i^2$ (with $1 \leq i \leq n$).

When $a = 0$, which occurs for the cases $(D, n) = (11, 4), (11, 7)$ and $(10, 5)$, the resulting solutions become AdS domain walls. The metrics in these cases become asymptotically-AdS spacetimes in seven, four and five dimensions. These AdS domain-wall solutions are sphere reductions of the decoupling limits of ellipsoidal distributions of M-branes and D3-branes. These cases (and subsets) were studied previously in [8, 9, 10, 11, 12, 13].

In this paper, we shall extend the previous analysis to include the cases where the dilaton-coupling constant a is non-vanishing. For these cases, the domain-wall metric (24) is no longer asymptotically AdS, but instead is asymptotic to a vacuum domain wall as $r \to \infty$, given by

$$ds_d^2 = \rho^{\frac{4(n+1)}{a^2(D-2)}}\, dx^\mu\, dx_\mu + g^{-2}\, d\rho^2\,. \tag{29}$$

where $\rho \sim (gr)^{\frac{a^2(D-2)}{4(d-2)}}$. This metric is flat as ρ approaches at infinity.

In the region near $r = 0$, the metric structure depends on the number of non-vanishing parameters ℓ_i. If k of the ℓ_i are non-vanishing, we have

$$ds_d^2 = \rho^\gamma\, dx^\mu dx_\mu + d\rho^2\,, \tag{30}$$

where

$$\gamma = \frac{4(n+1-k)}{a^2(D-2) + 2(d-3)k}\,, \qquad \rho = (gr)^{\frac{a^2(D-2)+2(d-3)k}{4(d-2)}}\,. \tag{31}$$

Thus we see that at $r = 0 = \rho$, the solution is generic singular. To see if the singularity is naked or not, we evaluate

$$\gamma - 2 = \frac{4(d-2)(n-3-k)}{a^2(D-2) + 2(d-3)k}\,. \tag{32}$$

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Thus for $n = 0, 1, 2, 3$, the solution has a naked singularity for all values of k. For $n \geq 4$, the singularity is naked for $k > n - 3$, but marginal for $k \leq n - 3$.

If we oxidise the solution back to D dimensions, it acquires an interpretation as a continuous distribution of $(D - n - 2)$-branes, given by

$$
\begin{aligned}
ds_D^2 &= H^{-\frac{n-1}{D-2}} dx^\mu dx_\mu + H^{\frac{D-n-1}{D-2}} dy^m dy^m \,, \\
e^{-\frac{2}{a}\phi} &= H \,. \qquad F_{(n)} = e^{-a\phi} \hat{*} (d^{D-n-1} x \wedge dH^{-1}) \,.
\end{aligned}
\tag{33}
$$

where H and the transverse Euclidean metric are given by

$$
\begin{aligned}
dy^m dy^m &= h^{-\frac{1}{2}} \widetilde{\Delta} \, dr^2 + r^2 \sum H_i \, d\mu_i^2 \,, \\
H &= \frac{1}{(gr)^{n-1} \widetilde{\Delta}} \,, \qquad \widetilde{\Delta} = h^{\frac{1}{2}} \sum \frac{\mu_i^2}{H_i} \,.
\end{aligned}
\tag{34}
$$

The function H is a harmonic function of the Euclidean transverse space, and it can be expressed as

$$
H = g^{-(n-1)} \int \frac{\sigma(\vec{y}') \, d^{n+1} y'}{|\vec{y} - \vec{y}'|^{n-1}} \,,
\tag{35}
$$

where $\sigma(\vec{y})$ is the distribution function. The harmonic functions in our cases here are associated with ellipsoidal distributions.

A detailed analysis is given in [12], where the charge-distribution functions are obtained in the non-dilatonic cases of 3-branes in $D = 10$, and M2-branes and M5-branes in $D = 11$. The analysis here is almost identical, and we shall not enumerate all the possibilities. It was observed in [12] that although the results for the charge-distribution functions are distinctly different depending upon how many of the ℓ_i parameters are non-zero, by carefully taking limits in which some of the parameters are sent to zero one can view them all as being derived from a maximally-degenerate case with all $(n + 1)$ parameters non-zero. The distribution function with all the ℓ_i non-vanishing is given by [12]

$$
\sigma_{n+1} = \frac{1}{V_n \prod_{i=1}^{n+1} \ell_i} \delta'(1 - \sum_{i=1}^{n+1} \frac{y_i^2}{\ell_i^2}) \,,
\tag{36}
$$

where V_n is the volume of the n-sphere and $'$ refers to the derivative with respect to the δ-function argument. This same charge distribution arises in our present cases, too.

As an example, let us consider what happens if one of the parameters, say ℓ_{n+1} is sent to zero. It is clear from (36) that the integration in (35) over the associated direction y'_{n+1} will become dominated by the contribution from y'_{n+1} close to zero, and so the $(n+1)$-parameter charge distribution σ_{n+1} in the $\ell_{n+1} \longrightarrow 0$ limit will become the n-parameter distribution

$$
\sigma_n(y_1, \ldots, y_{n+1}) = \delta(y_{n+1}) \int_{-\infty}^{\infty} d\tilde{y}_{n+1} \, \sigma_{n+1}(y_1, \ldots, y_n, \tilde{y}_{n+1}) \,.
\tag{37}
$$

Evaluating the integral, we obtain

$$\sigma_n = \frac{1}{2\,V_n\,\prod_{i=1}^n \ell_i}\left(2(1-\sum_{i=1}^n \frac{y_i^2}{\ell_i^2})^{-1/2}\,\delta(1-\sum_{i=1}^n \frac{y_i^2}{\ell_i^2})-(1-\sum_{i=1}^n \frac{y_i^2}{\ell_i^2})^{-3/2}\,\Theta(1-\sum_{i=1}^n \frac{y_i^2}{\ell_i^2})\right)\delta(y_{n+1})\,. \tag{38}$$

Sending another parameter, say ℓ_n to zero, we next obtain the $(n-1)$-parameter charge distribution

$$\sigma_{n-1} = \frac{\pi}{V_n\,\prod_{i=1}^{n-1}\ell_i}\,\delta(1-\sum_{i=1}^{n-1}\frac{y_i^2}{\ell_i^2})\,\delta^{(2)}(y_n,y_{n+1})\,. \tag{39}$$

Further details of the successive results for smaller numbers k of non-vanishing parameters ℓ_i are given in [12]. Note that the distributions associated with $k = n+1$ and $k = n$ non-vanishing ℓ_i parameters both have regions with negative as well as positive p-brane tensions. For $k \le n-1$, on the other hand, the distributions contain only positive tensions.

When all the parameters ℓ_i vanish, corresponding to the "vacuum" domain-wall solution in $d = D-n$ dimensions, the D-dimensional solution describes coincident $(D-n-2)$-branes at the origin, with the constant 1 in the harmonic function H dropped. This can be viewed as a certain decoupling limit. The metric of the solution in the Einstein frame can then be expressed as

$$ds_{\rm E}^2 = e^{\frac{a}{n-1}\phi}\left(r^{n-3}\,dx^\mu dx_\mu + \frac{dr^2}{r^2} + d\Omega_n^2\right)\,. \tag{40}$$

One can then define a dual frame $ds_{\rm dual}^2 = e^{-a\phi/(n-1)}\,ds_{\rm E}^2$, in which the Lagrangian becomes

$$\mathcal{L} = e\,e^{\frac{a(D-2)}{2(n-1)}\phi}\left(R + \frac{(D-2)(n^2-n\,D-n+3D-2)}{2(n-1)^2}\,(\partial\phi)^2 - \frac{1}{2n!}\,F_{(n)}^2\right)\,. \tag{41}$$

In this dual frame, the metric is AdS$\times S^n$ if $n \ne 3$, and Minkowski$\times S^3$ when $n = 3$. This analysis was given in detail in [14] for $D = 10$, leading to the conjecture of a Domain-wall/QFT correspondence. Further studies of the Domain-wall/QFT correspondence in general dimensions were given in [15].

It is of interest to note that in the dual frame, the metric depends only on the dimension n of the internal sphere, but it is independent of D; the D-dependence of the Einstein-frame metric can all extracted as a conformal factor. Note that the dual frame metric has qualitative differences in the three situations $n > 3$, $n = 3$ and $n < 3$. For $n = 3$, the dual frame is Minkowskian, whilst for $n \ne 3$, the spacetime is AdS. However, for $n > 3$ we have that $r = 0$ is the horizon, whilst for $n < 3$ the horizon is instead at $r = \infty$. These qualitative differences have significance for the structure of the spectrum in the dual QFT, which we shall analyse in the next section.

When the ℓ_i parameters are non-vanishing, the metric of the distributed branes in the

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dual frame is given by

$$ds_{\text{dual}}^2 = \widetilde{\Delta}^{\frac{n-3}{n-1}} \, ds_d^2 + g^{-2} \, \widetilde{\Delta}^{\frac{2}{n-1}} \sum H_i^2 \, d\mu_i^2 \,, \tag{42}$$

where

$$ds_d^2 = (gr)^{n-3} \, dx^\mu dx_\mu + \frac{dr^2}{(gr)^2 \, h^{\frac{1}{2}}} \,. \tag{43}$$

Again we see that the metric does not manifestly depend on D, but on n instead.

4   Analysis of the spectrum

A minimally-coupled scalar field Φ obeys the wave equation

$$\partial_\mu(\sqrt{-g} \, g^{\mu\nu} \, \partial_\nu \Phi) = 0 \,. \tag{44}$$

We make the Ansatz $\Phi = e^{ip \cdot x} \, \chi(r)$, where $m^2 = -p \cdot p$ determines the mass of the fluctuating mode, and so the wave equation has the following general form

$$r^{-1} \partial_r \Big[r^{-1} \prod_{i=1}^{n+1} \sqrt{r^2 + \ell_i^2} \, \partial_r \chi \Big] = -\mathcal{Q} \chi \,, \tag{45}$$

where $\mathcal{Q} = m^2 \, g^{-\frac{1}{2}(n+1)}$. Remarkably, the wave equation depends only on the dimension of the internal sphere, but otherwise is independent of details of the original higher-dimensional theory.

It is helpful to cast the wave equation into the Schrödinger form, which can be done by first writing the metric in a manifestly conformally-flat frame as

$$ds^2 = e^{2A(z)} \, (dx^\mu \, dx_\mu + dz^2) \,, \tag{46}$$

by means of an appropriate coordinate transformation. The coordinate z runs from 0 to z^*, and $A(z)$ has the following asymptotic behaviour:

$$e^{2A} \sim (z - z^*)^{\gamma^*} \,, \qquad \gamma^* = -\frac{2(n+1)}{(d-2)(n-3)} \quad \text{for} \quad z \to z^* \,,$$

$$e^{2A} \sim z^{\tilde{\gamma}} \,, \qquad \tilde{\gamma} = \frac{2\gamma}{2-\gamma} = -\frac{2(n+1-k)}{(d-2)(n-3-k)} \,, \quad \text{for} \quad z \to 0 \,. \tag{47}$$

Making the field redefinition $\chi = e^{-(D-2)A/2} \, \psi$, the wave equation assumes the form

$$(-\partial^2 - V) \, \psi = \tfrac{1}{4} \mathcal{Q} \, \psi \,, \tag{48}$$

with the Schrödinger potential given by

$$V = \frac{d-2}{2} \, A'' + \frac{(d-2)^2}{4} \, (A')^2 \,. \tag{49}$$

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The asymptotic behavior of the potential is given by

$$V \sim \frac{c^*}{(z - z^*)^2}, \qquad \text{for} \qquad z \to z^*,$$

$$V \sim \frac{c}{(z - \tilde{z})^2}, \qquad \text{for} \qquad z \to \tilde{z}, \tag{50}$$

where

$$c^* = -\tfrac{1}{4} + \frac{(n-1)^2}{(n-3)^2}, \qquad c = -\tfrac{1}{4} + \frac{(n-1-k)^2}{(n-3-k)^2} \geq -\tfrac{1}{4}. \tag{51}$$

The range of the coordinate z is determined by the values of z^* and $\tilde{z}$, which in the original coordinate r correspond to $r \to \infty$ and $r \to 0$ limit, respectively. It is understood that if z^* or $\tilde{z}$ equals $\pm\infty$, the potential in (50) is of the form $\pm 1/z^2$.

Note that for $n \leq 3$ $[n \geq 4]$ the limit $r \to \infty$ corresponds to $z \to z^*$ with $z^* = \infty$ $[z^* = \text{finite}]$. On the other hand for $n - k \leq 3$ $[n - k \geq 4]$ the limit $r \to 0$ corresponds to $z \to \tilde{z}$ where $\tilde{z} = 0$ $[\tilde{z} = -\infty]$. When $n = 3$ or $k = n - 3$, where the denominator of the above expression vanishes, the coordinate z depends logarithmically on the original coordinate r $(z \sim \log(r))$ and the Schrödinger potential becomes constant: $V = 1/4$.

Note that since the wave equation is independent of D, whilst the metric depends on D, it may be more instructive to perform a field redefinition directly on the wave equation (45). This can be done by first defining $y = r^2$, and then introducing a new coordinate z defined by $\partial y / \partial z = \sqrt{f(y)}$, where $f(y) = [\prod_{i=1}^{n+1}(y + \ell_i^2)]^{1/2}$. (These are the defining equations that relate z and r coordinates.) The Schrödinger potential is then given by [12]

$$V = \tfrac{1}{4}\partial_z^2 \log f + \tfrac{1}{16}(\partial_z \log f)^2. \tag{52}$$

and it clearly depends on n and ℓ_i $(i = 1, \cdots, k)$ only.

4.1   Vacuum excitations

When all the parameters ℓ_i vanish, the solution (24) becomes a domain-wall vacuum solution. In the case when $a^2 = 0$, which occurs for $(D, n) = (11, 7)$, $(10, 5)$ and $(11, 4)$, the solution is just the AdS spacetime in $d = 4$, 5 and 7 respectively. For $a^2 \neq 0$, the metric of the solution is (30). The metric is flat near $\rho = \infty$, but becomes singular as ρ approaches zero. Since we have

$$\frac{4(n+1)}{a^2(D-2)} - 2 = \frac{4(d-2)(n-3)}{a^2(D-2)}, \tag{53}$$

the singularity is marginal for $n \geq 3$, but naked for $n < 3$.

The characteristics of the Schrödinger potential depend only the value of n. For $n = 0, 1, 2$, the potential is given by

$$V = \frac{c^*}{z^2}, \tag{54}$$

where c^* is given in (51). The coordinate z runs from 0 to infinity as r runs from 0 to infinity. For $n = 3$, the potential is a constant, $V = 1/4$, and the coordinate z runs from minus infinity to infinity as r runs from 0 to infinity. For $n \geq 4$, the potential is of the same form as (54), but the coordinate z now runs from minus infinity to 0 as r runs from 0 to infinity. Thus we see that although the domain-wall vacuum can have (naked) singularities, the quantum fluctuations are nevertheless well behaved. In fact it is straightforward to solve the minimally-coupled scalar wave equation in the domain-wall vacuum, namely

$$r^{-1}\partial_r(r^n\,\partial_r\chi) = -\mathcal{Q}\,\chi\,. \tag{55}$$

If we define a new dependent variable y by

$$\chi(r) = y(r)\,r^{-(n-1)/2}\,, \tag{56}$$

and change to the new independent variable z defined by

$$z = \frac{2\sqrt{\mathcal{Q}}}{n-3}\,r^{-(n-3)/2}\,, \tag{57}$$

then the wave equation (55) becomes the Bessel equation

$$z^2\,y''(z) + z\,y'(z) + (z^2 - \nu^2)\,y(z) = 0\,, \tag{58}$$

where

$$\nu = \frac{n-1}{n-3}\,. \tag{59}$$

The solutions to (55) are therefore given by

$$\chi(r) = a\,r^{-(n-1)/2}\,J_\nu\Big(\frac{2\sqrt{\mathcal{Q}}}{n-3}\,r^{-(n-3)/2}\Big) + b\,r^{-(n-1)/2}\,Y_\nu\Big(\frac{2\sqrt{\mathcal{Q}}}{n-3}\,r^{-(n-3)/2}\Big)\,. \tag{60}$$

A special case arises for $n = 3$ (the Schrödinger potential is constant, $V = 1/4$, there) for which we find

$$\chi(r) = a\,r^{-1+\mathrm{i}\,\sqrt{\mathcal{Q}-1}} + b\,r^{-1-\mathrm{i}\,\sqrt{\mathcal{Q}-1}}\,. \tag{61}$$

The requirement that $Q \geq 1$ corresponds to the condition that there is an energy gap.

4.2   Domain-wall excitations

When some of the ℓ_i parameters are non-vanishing, the wave equations cannot in general be solved explicitly. Here, for simplicity, we shall consider the case where all the non-vanishing ℓ_i are equal. There are certain examples where the wave equations can be solved exactly. Two of these ($n = 5$, $k = 2$ with two equal charges ℓ_i, and $n = 5$, $k = 4$ with four equal

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charges) are solved in [9]. Another solvable example is $n = 3$, $k = 2$, with the two non-vanishing charges equal, say $\ell_1 = \ell_2 \equiv \ell$. In this case, if we let $x = -r^2/\ell^2$, equation (45) becomes the hypergeometric equation

$$x(1-x)\,\chi'' + (1-2x)\,\chi' - \tfrac{1}{4}\mathcal{Q}\chi = 0\,, \tag{62}$$

and so one solution gives

$$\chi_1 = {}_2F_1[a,b;1;-\frac{r^2}{\ell^2}]\,, \qquad a = \tfrac{1}{2} + \tfrac{i}{2}\sqrt{\mathcal{Q}-1}\,, \qquad b = \tfrac{1}{2} - \tfrac{i}{2}\sqrt{\mathcal{Q}-1}\,. \tag{63}$$

Note again that $\mathcal{Q} > 1$ corresponds to the condition that the (continuous) spectrum has a gap owing to the properties of the Schrödinger potential $V \geq 1/4$ (figure g). Note that at small r we therefore have $\chi_1 \sim 1$, while at large r the asymptotic behaviour is of the same form as in (61). Since the c argument of the hypergeometric function ${}_2F_1[a,b;c;x]$ in (63) is an integer, the second solution χ_2 of (62) must be obtained by taking an appropriate limit of the standard second solution $x^{1-c}\,{}_2F_1[a-c+1,b-c+1;2-c;x]$. This gives a logarithmic behaviour of the form $\chi_2 \sim \log r$ at small r.

For the remaining examples, although we cannot solve the wave equation analytically we can determine the structure of the spectra for the various cases from the forms of their Schrödinger potentials. The results are summarised in Table 1.

n	k	z-range	V type	Spectrum
0,1	0,1	$(0,\infty)$	a	continuous
2	0	$(0,\infty)$	b	continuous
	1,2	$(0,\infty)$	c	continuous
3	0	$(-\infty,\infty)$	$V = \tfrac{1}{4}$	cont. with gap
	1	$(0,\infty)$	d	disc., cont. with gap
	2,3	$(0,\infty)$	e	cont. with gap
≥ 4	$\leq n-4$	$(-\infty,0)$	f	continuous
	$n-2$	$(-\infty,0)$	g	cont. with gap
	$n-3$	$(-1,0)$	h	discrete
	$n, n-1$	$(-1,0)$	i	discrete

Table 1: Spectral analysis for domain-wall solutions for various n's and k's.

The various different types of structures of the potentials are sketched in Figure 1.
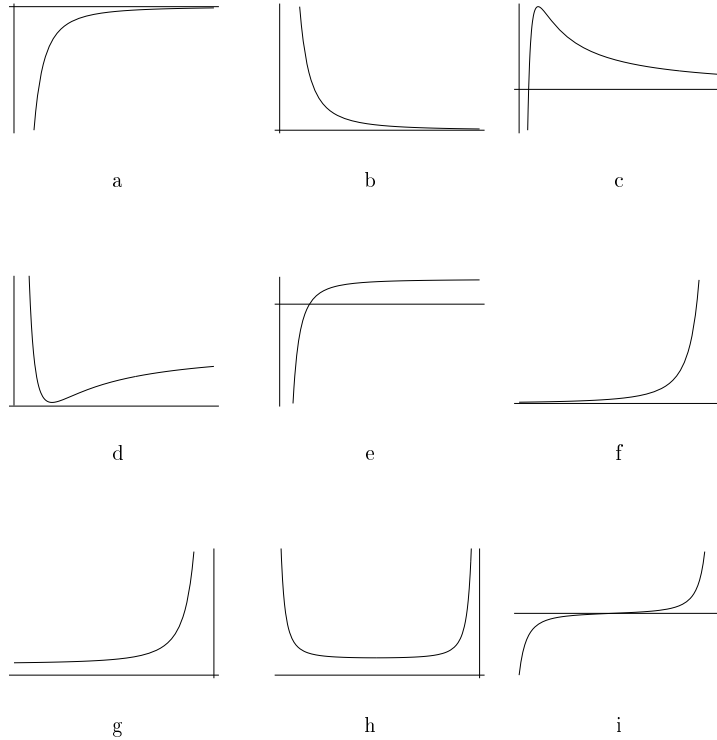
Figure 1: Sketches of the various Schrödinger potentials

5 Conclusions

In this paper we have studied consistent n-sphere reductions of a D-dimensional theory of gravity coupled to an n-form field-strength and a dilaton. Provided that the dilaton has a specific strength of coupling to the n-form, given by (3), we have proven the consistency of the non-linear Kaluza-Klein Ansatz for the n-sphere reduction in which there are n scalars parameterising right-ellipsoidal inhomogeneous deformations of the sphere.[2]

[2]We did not turn on the Kaluza-Klein gauge-fields in the reduction, which corresponds to a consistent truncation of the theory. However in the appendix we also discuss an n-sphere reduction of this Lagrangian that corresponds to making pair-wise identifications of the diagonal scalar fields, together with turning on the electric components of the Abelian Kaluza-Klein fields. This reduction provides a D-dimensional embedding of the $(D-n)$-dimensional non-extreme (large) charged-black holes as (near-extreme) spinning electric $(D-n-2)$-branes. In the BPS limit the charged black holes become neutral BPS domain-wall

The generality of these consistent reductions provides a framework within which we can address the D-dimensional embedding of a class of solutions of the reduced gauged supergravities in $d = D - n$ dimensions. In general, these gauged supergravities have potentials for the scalar fields that do not admit AdS ground-states, and thus in general, the typical solutions correspond to domain walls that are asymptotic to the "dilatonic" vacuum. In particular, we found the general class of BPS domain-wall solutions that are specified by $k = \{1, \cdots, n\}$ parameters, which characterise the harmonic functions of k non-trivial scalars.

All these solutions have explicit representations as continuous distributions of extremal $(D - n - 2)$-branes, and thus in the context of the Domain-wall/QFT correspondence they describe the Coulomb phase of the dual strongly-coupled field theory.

The universal properties of these gravity solutions manifest themselves in the properties of the wave equations in these backgrounds. For minimally-coupled scalars, the wave equations are completely universal and depend only on the dimension n of the compactifying sphere and the number k of parameters in the harmonic functions specifying the non-trivial scalar fields. Remarkably, the wave equations are *independent* of the original dimension D. Thus in the dual field theory the bound-state spectrum is completely specified by n and k. We gave an analysis of the spectra for all these cases.

One of the interesting outcomes of our study is the generality and universality of the BPS solutions for the specific subsector of the sphere-reduced gravity theories. This provides a strong indication that the dual field theories should exhibit the same intriguing features, irrespective of the dimension.

Acknowledgments

A   Single-charge rotating p-branes

The Lagrangian (2) also admits rotating p-brane solutions. In this appendix, we show that such a rotating p-brane associated with a given by (3) can be dimensionally reduced on the transverse spherical space, and it then gives rise to a domain-wall black hole in the lower dimension. The Lagrangian (2) admits an electric $(d - 1)$-brane with $d = n - 1$, or a

solutions.

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magnetic $(d-1)$-brane with $d = D - n - 1$. We shall consider only the magnetic solution here, since the electric one can be viewed as a magnetic solution of the dual $(D-n)$-form field strength $F_{(D-n)}$. There are two cases arising, depending on whether $\tilde{d}$ is even or odd.

Case 1: $n = 2N - 1$

In this case, there are N angular momenta ℓ_i, with $i = 1, 2, \ldots, N$. We find that the metric of the rotating $(n-2)$-brane solution to the equations following from (2) is [16]

$$
\begin{aligned}
ds_D^2 &= H^{-\frac{n-1}{D-2}}\Big(-(1 - \frac{2m}{r^{n-1}\Delta})\, dt^2 + d\vec{x} \cdot d\vec{x} \Big) + H^{\frac{D-n-1}{D-2}}\Big[\frac{\Delta\, dr^2}{H_1 \cdots H_N - \frac{2m}{r^{-(n-1)}}} \\
&\quad + r^2 \sum_{i=1}^N H_i(d\tilde{\mu}_i^2 + \tilde{\mu}_i^2\, d\phi_i^2) - \frac{4m \cosh\alpha}{r^{n-1} H \Delta}\, dt\, (\sum_{i=1}^N \ell_i\, \tilde{\mu}_i^2\, d\phi_i) \\
&\quad + \frac{2m}{r^{n-1} H \Delta}\, (\sum_{i=1}^N \ell_i\, \tilde{\mu}_i^2\, d\phi_i)^2 \Big]\ ,
\end{aligned}
\tag{64}
$$

where the functions Δ, H and H_i are given by

$$
\Delta = H_1 \cdots H_N \sum_{i=1}^N \frac{\tilde{\mu}_i^2}{H_i}\ , \qquad H = 1 + \frac{2m \sinh^2\alpha}{r^{n-1}\Delta}\ ,
$$

$$
H_i = 1 + \frac{\ell_i^2}{r^2}\ , \qquad i = 1, 2, \ldots, N\ .
\tag{65}
$$

The dilaton ϕ and the field strength $F_{(n)}$ are given by

$$
e^{2\phi/a} = H\ , \qquad e^{a\phi}\ast F_{(n)} = \frac{dH^{-1}}{\sinh\alpha} \wedge \Big(\cosh\alpha\, dt + \sum_{i=1}^N \ell_i\, \mu_i^2\, d\phi_i \Big) \wedge d^{D-n-2}x\ .
\tag{66}
$$

The N quantities $\tilde{\mu}_i$ are subject to the constraint $\sum_i \tilde{\mu}_i^2 = 1$. They are related to our previous coordinates constrained μ_i on the sphere as follows:

$$
\mu_1 + \mathrm{i}\,\mu_2 = \tilde{\mu}_1\, e^{\mathrm{i}\,\phi_1}\ , \qquad \mu_3 + \mathrm{i}\,\mu_4 = \tilde{\mu}_2\, e^{\mathrm{i}\,\phi_2}\ , \qquad etc.
\tag{67}
$$

We now consider the decoupling limit, which is obtained by making the rescalings

$$
m \to \epsilon^{n-1}\, m\ , \qquad \sinh\alpha \to \epsilon^{-(n-1)/2}\, \sinh\alpha\ ,
$$

$$
r \to \epsilon\, r\ , \qquad x^\mu \to \epsilon^{-2}\, x^\mu\ , \qquad \ell_i \to \epsilon\, \ell_i
\tag{68}
$$

and then sending $\epsilon \to 0$. In this limit, the additive constant 1 in the function H in (65) can be dropped. Furthermore, the last term in (64) can also be dropped. The remaining metric can be expressed as

$$
ds_D^2 = Y^{\frac{2}{D-2}}\Big(\Delta^{\frac{n-1}{D-2}}\, ds_d^2 + g^{-2}\, \Delta^{-\frac{(D-n-1)}{(D-2)}} \sum_{i=1}^N \bar{X}_i^{-1}(d\tilde{\mu}_i^2 + \tilde{\mu}_i^2(d\phi_i + g\, A_{(1)}^i)^2) \Big)\ ,
\tag{69}
$$

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where $\Delta = \sum \bar{X}_i \, \tilde{\mu}_i^2$, and $g = (2m \, \sinh^2 \alpha)^{-1/(n-1)}$. The $d = D - n$ dimensional metric and the scalar fields X_i are given by

$$
\begin{aligned}
ds_d^2 &= -h^{-\frac{d-3}{d-2}} f \, dt^2 + h^{\frac{1}{d-2}} \Big(\frac{dr^2}{(gr)^{5-n} f} + (gr)^{n-3} d\vec{x} \cdot d\vec{x} \Big), \\
X_i &= (gr)^{\frac{a^2 \, (D-2)}{4(d-2)}} \, h^{\frac{(d-3)}{2(d-2)}} \, H_i^{-1}, \\
Y &= \prod_{i=1}^{N} X_i = \Big((gr)^{n+1} \, h^{-1} \Big)^{\frac{a^2 \, (D-2)}{8(d-2)}}, \\
A_{(1)}^i &= \frac{1 - H_i^{-1}}{g \, \ell_i \, \sinh \alpha} \, dt, \qquad h = \prod_{i=1}^{N} H_i. \\
f &= (gr)^{n-3} (h - \frac{2m}{r^{n-1}}).
\end{aligned}
\tag{70}
$$

This solution describes N electrically-charged black holes in a d-dimensional domain-wall background.

Note that in general, the abstract metric Ansatz that we have written in (69) does *not* (at least as it stands) correspond to part of a consistent Kaluza-Klein reduction. It can be viewed as a modification of the general consistent metric Ansatz in (6) in which we first (consistently) partially truncate the scalars by setting them equal in pairs ($X_1 = X_2 = \bar{X}_1$, $X_3 = X_4 = \bar{X}_2$, *etc*). Then, having also made the redefinitions (67), we introduce a $U(1)$ gauge field $A_{(1)}^i$ associated with the rotation in each of the original 2-planes (μ_1, μ_2), (μ_3, μ_4), *etc.* (Although we have not presented it here, we can also straightforwardly carry out the same steps on the original Ansätze for $\hat{\phi}$ and $\hat{F}_{(n)}$ in (6) too.) This *does* give a consistent reduction in the case $(10, 5)$ discussed in [16], but in general additional fields would have to be included too. The reason for this is that the $U(1)$ gauge fields, in quadratic products of the form $F_{(2)}^i \wedge F_{(2)}^j$, will act as sources for other fields. In the special case of $(D, n) = (10, 5)$, they actually act as sources for themselves (corresponding to cubic Chern-Simons terms in the five-dimensional theory), but in the other cases they will act as sources for additional fields, requiring a larger set of fields in the Kaluza-Klein reduction Ansatz.

The metric (69), together with analogously-obtained expressions for $\hat{\phi}$ and $\hat{F}_{(n)}$, is nevertheless still usable in appropriate circumstances. The problematic terms $F_{(2)}^i \wedge F_{(2)}^j$ actually vanish for our specific domain-wall black hole solutions since all the $U(1)$ charges are purely electric. This means that these particular lower-dimensional configurations will lift to the higher dimension without necessitating the turning-on of the additional fields that would be needed for a fully-consistent Ansatz, but which have been omitted in our discussion. Thus we still have an exact embedding of these specific solutions in the higher dimension.

Here, the solution has the same form as (64), but with the range of the index i extended to include 0. However, there is no angular momentum parameter or azimuthal coordinate associated with the extra index value, and so $\ell_0 = 0$ and $\phi_0 = 0$. The $\tilde{\mu}_i$ and ϕ_i coordinates are now related to the original coordinates μ_i on the sphere by

$$\mu_0 = \tilde{\mu}_0\,, \qquad \mu_1 + \mathrm{i}\,\mu_2 = \tilde{\mu}_1\,e^{\mathrm{i}\,\phi_1}\,, \qquad \mu_3 + \mathrm{i}\,\mu_4 = \tilde{\mu}_2\,e^{\mathrm{i}\,\phi_2}\,, \qquad etc. \tag{71}$$

Otherwise, all the formulae in Case 1 generalise to this case, simply by extending the summation to span the range $0 \le i \le N$. Of course $H_0 = 1$ as a consequence of $\ell_0 = 0$.

Note that for $a = 0$, we have $(D, n) = (11, 7), (11, 4)$ and $(10, 5)$. These correspond to the rotating M-branes [17, 16] and D3-branes [8, 18, 16].

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